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SYMMETRIC IDENTITIES OF HIGHER-ORDER DEGENERATE q-BERNOULLI POLYNOMIALS

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Abstract. The purpose of this paper is to give symmetric identities for higher-order degenerate q-Bernoulli polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$.

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. The q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let q be an indeterminate such that $|1-q|_p < p^{-\frac{1}{p-1}}$. As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \text{(see [7, 9, 10])}. \tag{1.1}$$

When $x = 0$, $B_n = B_n(0)$ are called ordinary Bernoulli numbers. From (1.1), we can derive the following recurrence relation related to B_n as follows:

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{1.2}$$

with the usual convention about replacing B^n by B_n. Let $UD(\mathbb{Z}_p)$ be the space of all uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is given by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \quad \text{(see [8, 10])}. \tag{1.3}$$

From (1.3), we note that

$$I_0(f_1) - I_0(f) = f'(0), \quad \text{(see [8, 10])}, \tag{1.4}$$

where $f_1(x) = f(x+1)$, $f'(0) = \frac{d}{dx}f(x)\big|_{x=0}$. By (1.4), we easily get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.5}$$

Thus, from (1.5), we have

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y) = B_n(x), \quad (n \geq 0). \tag{1.6}$$

For $r \in \mathbb{N}$, the higher-order Bernoulli polynomials are defined by the multivariate p-adic invariant integrals on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r+x)t} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{t}{e^t - 1}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [7, 13])}. \end{aligned} \tag{1.7}$$

Comparing the coefficients on the both sides of (1.7), we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_r + x)^n d\mu_0(x_1) \cdots d\mu_0(x_r) = B_n^{(r)}(x), \tag{1.8}$$

where $B_n^{(r)}(x)$ are called higher-order Bernoulli polynomials. In [2], L. Carlitz considered q-Bernoulli numbers which are given by

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{1.9}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$. Note that $\lim_{q \to 1} \beta_{n,q} = B_n$, $(n \geq 0)$. He also defined q-Bernoulli polynomials as follows:

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}, \quad \text{(see [2, 13])}. \tag{1.10}$$

For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N - 1} f(x) q^x, \quad \text{(see [13])}. \tag{1.11}$$

In [13], Kim proved that Carlitz's q-Bernoulli polynomials can be represented by the p-adic q-integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}. \tag{1.12}$$

L. Carlitz considered the degenerate Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}^*(x) \frac{t^n}{n!}, \quad \text{(see [3])}. \tag{1.13}$$

When $x = 0$, $B_{n,\lambda}^* = B_{n,\lambda}^*(0)$ are called degenerate Bernoulli numbers. Note that $\lim_{\lambda \to 0} B_{n,\lambda}^*(x) = B_n(x),\ (n \geq 0)$.

Recently, Kim defined (fully) degenerate Bernoulli polynomials which are different from Carlitz's degenerate Bernoulli polynomials:

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_0(y) &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} \quad \text{(see [17])}. \end{aligned} \tag{1.14}$$

Note that $\lim_{\lambda \to 0} B_{n,\lambda}(x) = B_n(x),\ (n \geq 0)$. Indeed,

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\lambda}^*(x) \frac{t^n}{n!} &= \frac{t}{\log(1+\lambda t)^{\frac{1}{\lambda}}} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_0(y) \\ &= \left(\sum_{l=0}^{\infty} b_l \frac{\lambda^l}{l!} t^l \right) \left(\sum_{m=0}^{\infty} B_{m,\lambda}(x) \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \lambda^l b_l B_{n-l,\lambda}(x) \right) \frac{t^n}{n!}, \end{aligned} \tag{1.15}$$

where b_n is the n-th Bernoulli numbers of the second kind. Thus, by (1.15), we get

$$B_{n,\lambda}^*(x) = \sum_{l=0}^{n} \binom{n}{l} \lambda^l b_l B_{n-l,\lambda}(x),\ (n \geq 0). \tag{1.16}$$

The higher-order Carlitz's q-Bernoulli polynomials are given by the multivariate p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{[x_1+x_2+\cdots+x_r+x]_q t} d\mu_q(x_1)\cdots d\mu_q(x_r)\\&=\sum_{n=0}^{\infty}\beta_{n,q}^{(r)}(x)\frac{t^n}{n!},\quad \text{(see [13])}.\end{aligned}\tag{1.17}$$

Note that $\lim_{q\to 1}\beta_{n,q}^{(r)}(x)=B_n^{(r)}(x),\ (n\geq 0)$. Recently, Kim introduced higher-order degenerate q-Bernoulli polynomials which are derived from the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[x_1+x_2+\cdots+x_r+x]_q} d\mu_q(x_1)\cdots d\mu_q(x_r)\\&=\sum_{n=0}^{\infty}\beta_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!},\quad \text{(see [17])}.\end{aligned}\tag{1.18}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda,q}^{(r)}(x)=\beta_{n,q}^{(r)}(x)$ and $\lim_{q\to 1}\beta_{n,\lambda,q}^{(r)}(x)=B_{n,\lambda}^{(r)}(x)$, where $B_{n,\lambda}^{(r)}(x)$ are the higher-order degenerate Bernoulli polynomials which are given by the generating function to be

$$\left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\quad \text{(see [3])}.\tag{1.19}$$

The purpose of this paper is to give some identities of symmetry for the higher-order degenerate q-Bernoulli polynomials which are derived from the integral equations of the p-adic q-integral on $\mathbb{Z}_p$.

2. Symmetric identities for the higher-order degenerate q-Bernoulli polynomials

Let $\lambda,t\in\mathbb{C}_p$ with $|\lambda|_p\leq 1$ and $|t|_p<p^{-\frac{1}{p-1}}$ and let $w_1,w_2\in\mathbb{N}$. From (1.4) and (1.19), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}} d\mu_0(x_1)\cdots d\mu_0(x_r)&=\left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.1}$$

In view of (1.17), we consider the higher-order degenerate q-Bernoulli polynomials which are derived from the multivariate p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[x_1+x_2+\cdots+x_r+x]_q}d\mu_q(x_1)\cdots d\mu_q(x_r)=\sum_{n=0}^{\infty}\beta_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.2}$$

Thus, we note that $\lim_{q\to 1}\beta_{n,\lambda,q}^{(r)}(x)=B_{n,\lambda}^{(r)}(x)$, $(n\geq 0)$. Now, we observe that

$$\begin{aligned}(1+\lambda t)^{\frac{1}{\lambda}[x_1+x_2+\cdots+x_r+x]_q}&=\sum_{n=0}^{\infty}\binom{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q}{n}\lambda^n t^n\\ &=\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}[x_1+\cdots+x_r+x]_q\right)_n\frac{\lambda^n t^n}{n!}\\ &=\sum_{n=0}^{\infty}[x_1+x_2+\cdots+x_r+x]_q\cdots\Big([x_1+\cdots+x_r+x]_q-(n-1)\lambda\Big)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\Big([x_1+\cdots+x_r+x]_q\Big)_{n,\lambda}\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

where $\Big([x]_q\Big)_{0,\lambda}=1$, $\Big([x]_q\Big)_{n,\lambda}=[x]_q\Big([x]_q-\lambda\Big)\cdots\Big([x]_q-(n-1)\lambda\Big)$, $(n\geq 1)$. It is not difficult to show that

$$\Big([x_1+\cdots+x_r+x]_q\Big)_{n,\lambda}=\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}[x_1+\cdots+x_r+x]_q^l, \tag{2.4}$$

where $S_1(n,l)$ is the stirling number of the first kind. By (1.17) and (2.2), we get

$$\begin{aligned}\beta_{n,\lambda,q}^{(r)}(x)&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Big([x_1+\cdots+x_r+x]_q\Big)_{n,\lambda}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &=\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^l d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &=\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\beta_{l,q}^{(r)}(x).\end{aligned} \tag{2.5}$$

Let $w_1, w_2 \in \mathbb{N}$. Then, we have

$$\begin{aligned}
&\frac{1}{[w_1]_q^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q}{\lambda}}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\
&=\frac{1}{[w_1]_q^r}\lim_{N\to\infty}\sum_{y_1,\cdots,y_r=0}^{p^N-1}\frac{1}{[p^N]_{q^{w_1}}^r}(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q}{\lambda}}q^{w_1(y_1+\cdots y_r)}\\
&=\frac{1}{[w_1]_q^r}\lim_{N\to\infty}\frac{1}{[w_2p^N]_{q^{w_1}}^r}\sum_{y_1,\cdots,y_r=0}^{w_2p^N-1}(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q}{\lambda}}\\
&\quad\times q^{w_1y_1+w_1y_2+\cdots+w_1y_r}\\
&=\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_q^r}\sum_{i_1,\cdots,i_r=0}^{w_2-1}\sum_{y_1,\cdots,y_r=0}^{p^N-1}(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r (i_l+w_2y_l)]_q}{\lambda}}\\
&\quad\times q^{w_1(i_1+w_2y_1)+w_1(i_2+w_2y_2)+\cdots+w_1(i_r+w_2y_r)}\\
&=\sum_{i_1,\cdots,i_r=0}^{w_2-1}q^{w_1\sum_{l=1}^r i_l}\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_q^r}\sum_{y_1,\cdots,y_r=0}^{p^N-1}q^{w_1w_2y_1+w_1w_2y_2+\cdots+w_1w_2y_r}\\
&\quad\times(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r (i_l+w_2y_l)]_q}{\lambda}}.
\end{aligned}\tag{2.6}$$

By (2.6), we get

$$\begin{aligned}
I(w_1,w_2)&=\frac{1}{[w_1]_q^r}\sum_{i_1,\cdots,i_r=0}^{w_1-1}q^{w_2\sum_{l=1}^r i_l}\\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q}{\lambda}}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_r}}(y_r)\\
&=\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_q^r}\sum_{i_1,\cdots,i_r=0}^{w_2-1}\sum_{j_1,\cdots,j_r=0}^{w_1-1}\sum_{y_1,\cdots,y_r=0}^{p^N-1}q^{w_1\sum_{l=1}^r i_l+w_2\sum_{l=1}^r j_l+w_1w_2\sum_{l=1}^r y_l}\\
&\quad\times(1+\lambda t)^{\frac{1}{\lambda}[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r y_l]_q}.
\end{aligned}\tag{2.7}$$

On the other hand,

$$
\begin{aligned}
I(w_2,w_1) = & \lim_{N\to\infty} \frac{1}{[w_1w_2p^N]_q^r} \sum_{i_1,\cdots,i_r=0}^{w_1-1} \sum_{j_1,\cdots,j_r=0}^{w_2-1} \\
& \times \sum_{y_1,\cdots,y_r=0}^{p^N-1} q^{w_1\sum_{l=1}^r j_l + w_2\sum_{l=1}^r i_l + w_1w_2\sum_{l=1}^r y_l} \\
& \times (1+\lambda t)^{\frac{1}{\lambda}[w_1w_2x + w_1\sum_{l=1}^r j_l + w_2\sum_{l=1}^r i_l + w_1w_2\sum_{l=1}^r y_l]_q}.
\end{aligned} \tag{2.8}
$$

From (2.7) and (2.8), we note that $I(w_1,w_2) = I(w_2,w_1)$. Therefore, we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2 \in \mathbb{N}$, we have*

$$
\begin{aligned}
& \frac{1}{[w_2]_q^r} \sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l} \\
& \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[w_1w_2x + w_1\sum_{l=1}^r j_l + w_2\sum_{l=1}^r y_l]_q} d\mu_{q^{w_2}}(y_1)\cdots d\mu_{q^{w_2}}(y_r) \\
& = \frac{1}{[w_1]_q^r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2\sum_{l=1}^r j_l} \\
& \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[w_1w_2x + w_2\sum_{l=1}^r j_l + w_1\sum_{l=1}^r y_l]_q} d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r).
\end{aligned}
$$

It is not difficult to show that

$$
\left[w_1w_2x + \sum_{l=1}^r j_l w_2 + \sum_{l=1}^r y_l w_1 \right]_q = [w_1]_q \left[w_2x + \frac{w_2}{w_1}\sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}. \tag{2.9}
$$

Thus, by (2.2) and (2.9), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[w_1w_2x+\sum_{l=1}^r j_lw_2+\sum_{l=1}^r y_lw_1]_q}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1]_q}{\lambda}[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l]_{q^{w_1}}}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\frac{\lambda}{[w_1]_q}[w_1]_qt\right)^{\frac{[w_1]_q}{\lambda}[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l]_{q^{w_1}}}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\
&=\sum_{n=0}^{\infty}\beta^{(r)}_{n,\frac{\lambda}{[w_1]_q},q^{w_1}}\Big(w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l\Big)[w_1]_q^n\frac{t^n}{n!}.
\end{aligned}\tag{2.10}$$

Therefore, by Theorem 1 and (2.10), we obtain the following theorem.

Theorem 2.2. *Let $n\geq 0$ and $w_1,w_2\in\mathbb{N}$. Then we have*

$$\begin{aligned}
&[w_1]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}q^{w_2\sum_{l=1}^r j_l}\beta^{(r)}_{n,\frac{\lambda}{[w_1]_q},q^{w_1}}\Big(w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l\Big)\\
&=[w_2]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_2-1}q^{w_1\sum_{l=1}^r j_l}\beta^{(r)}_{n,\frac{\lambda}{[w_2]_q},q^{w_2}}\Big(w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l\Big).
\end{aligned}$$

In particular, if we take $w_2=1$, then we have

$$[w_1]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}q^{\sum_{l=1}^r j_l}\beta^{(r)}_{n,\frac{\lambda}{[w_1]_q},q^{w_1}}\Big(x+\frac{1}{w_1}\sum_{l=1}^r j_l\Big)=\beta^{(r)}_{n,\lambda,q}(w_1x).\tag{2.11}$$

From (2.11), we note that

$$\begin{aligned}
\beta^{(r)}_{n,q}(w_1x)&=\lim_{\lambda\to 0}\beta^{(r)}_{n,\lambda,q}(w_1x)\\
&=\lim_{\lambda\to 0}[w_1]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}q^{\sum_{l=1}^r j_l}\beta^{(r)}_{n,\frac{\lambda}{[w_1]_q},q^{w_1}}\Big(x+\frac{1}{w_1}\sum_{l=1}^r j_l\Big)\\
&=[w_1]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}q^{\sum_{l=1}^r j_l}\beta^{(r)}_{n,q^{w_1}}\Big(x+\frac{1}{w_1}\sum_{l=1}^r j_l\Big).
\end{aligned}$$

From (2.5), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}\right)_{n,\lambda}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)$$
$$=\sum_{l=0}^{n}S_1(n,l)[w_1]_q^l\lambda^{n-l}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}^l d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)$$
$$=\sum_{l=0}^{n}S_1(n,l)[w_1]_q^l\lambda^{n-l}\sum_{i=0}^{l}\binom{l}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j_1+\cdots+j_r]_{q^{w_2}}^i q^{w_2(l-i)\sum_{k=1}^{r}j_k}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[w_2x+y_1+\cdots y_r]_{q^{w_1}}^{l-i}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)$$
$$=\sum_{l=0}^{n}\sum_{i=0}^{l}S_1(n,l)\lambda^{n-l}[w_1]_q^{l-i}[w_2]_q^i\binom{l}{i}[j_1+\cdots+j_r]_{q^{w_2}}^i q^{w_2(l-i)\sum_{k=1}^{r}j_k}\beta_{l-i,q^{w_1}}^{(r)}(w_2x). \tag{2.12}$$

By (2.12), we get

$$[w_1]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}q^{w_2\sum_{l=1}^{r}j_l}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}\right)_{n,\lambda}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)$$
$$=\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i}S_1(n,l)\lambda^{n-l}[w_1]_q^{n+l-r-i}[w_2]_q^i\beta_{l-i,q^{w_1}}^{(r)}(w_2x)$$
$$\times\sum_{j_1,\cdots,j_r=0}^{w_1-1}[j_1+\cdots+j_r]_{q^{w_2}}^i q^{w_2(l+1-i)\sum_{k=1}^{r}j_k}$$
$$=\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i}S_1(n,l)\lambda^{n-l}[w_1]_q^{n+l-r-i}[w_2]_q^i\beta_{l-i,q^{w_1}}^{(r)}(w_2x)T_{l+1,i}^{(r)}(w_1|q^{w_2}) \tag{2.13}$$

where

$$T_{n,i}^{(r)}(w|q)=\sum_{j_1,\cdots,j_r=0}^{w-1}q^{(n-i)\sum_{l=1}^{r}j_l}[j_1+\cdots+j_r]_q^i. \tag{2.14}$$

Therefore, by Theorem 2 and (2.14), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$, we have*

$$\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i}S_1(n,l)\lambda^{n-l}[w_1]_q^{n+l-r-i}[w_2]_q^i\beta_{l-i,q^{w_1}}^{(r)}(w_2x)T_{l+1,i}^{(r)}(w_1|q^{w_2})$$
$$=\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i}S_1(n,l)\lambda^{n-l}[w_2]_q^{n+l-r-i}[w_1]_q^i\beta_{l-i,q^{w_2}}^{(r)}(w_1x)T_{l+1,i}^{(r)}(w_2|q^{w_1}).$$

Let us take $\lambda \to 0$. Then, we have

$$\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^{2n-r-i}[w_2]_q^i\beta_{n-i,q^{w_1}}^{(r)}(w_2x)T_{n+1,i}^{(r)}(w_1|q^{w_2})$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^{2n-r-i}[w_1]_q^i\beta_{n-i,q^{w_2}}^{(r)}(w_1x)T_{n+1,i}^{(r)}(w_2|q^{w_1}).$$

Remark Recently, several authors have studied the symmetry identities of Bernoulli and Euler polynomials and Carlitz's q-Bernoulli and q-Euler polynomials (see [1-24]).

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